Note on Static Game with Incomplete Information

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1 Information and Knowledge

This section is based upon reading of Levin, 2006.

1.1 A Model of Knowledge

Definition 1.1 (Model of Knowledge)

1. *A set of states Ω, one of which is true.*
2. *For each state $\omega \in \Omega$, and a given agent i, there is a set of information function $h_i(\omega)$.*
3. *An event is a set of states $E \subseteq \Omega$.*
4. *An agent knows E if E obtains at all the states that the agent believes are possible.*

Definition 1.2 (Information function)

Given a set of states Ω, an information function associates every state $\omega \in \Omega$ with a nonempty subset $h(\omega)$ of Ω.

Note on $h(\omega)$ *$h(\omega)$ is the set of states the agent believes to be possible at ω.*

Definition 1.3 (Partitional information function 1)

An information function is partitional if there is some partition of Ω such that for any $\omega \in \Omega$, $h(\omega)$ is the lement of the partition that contians ω.

Definition 1.4 (Partitional information function 2)

An information function is partitional iff it satisfies P1 and P2.

P1 $\omega \in P(\omega)$ *for every* $\omega \in \Omega$.

P2 *If* $\omega' \in P(\omega)$ *then* $P(\omega') = P(\omega)$.

Note on Interpretation

1. *Property P1 says that, given state ω, the agent is not convinced that the state is not ω.*
2. *Property P2 says that if ω' is also deemed possible, then the set of states that would be deemed possible were the state actually ω' must be the same as those currently deemed possible at ω.*

Example 0.1 Given state $\Omega = \{a, b\}$, the information funciton $P(a) = \{a\}, P(b) = \{a, b\}$ satisfies P1 but not P2.

Definition 1.5 (Knowledge function 1)

The agent's knowledge function of event E is the set of states at which the agent knows E:

$$K(E) = \{\omega \in \Omega : h(\omega) \subseteq E\}.$$

Note on Interpretation *If $h(\omega) \subset E$, then in state ω, the agent views $\neg E$ as impossible. Hence we say that the agent knows E.*

Example 0.2 Suppose $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, H = \{\{\omega_1, \omega_2\}, \{\omega_3\} \{\omega_4\}\}$, then $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$ and $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

Lemma 1.1 (Knowledge function's property)

The knowledge function derived from any information function satisfies:

- *K1 (Axiom of Awareness) $K(\Omega) = \Omega$.*
- *K2 $K(E) \cap K(F) = K(E \cap F)$.*
- *K3 If $E \subseteq F$, then $K(E) \subseteq K(F)$.*

If P1 is satisfied, then

- *K4 (Axiom of knowledge) $K(E) \subseteq E$.*

If $P(.)$ is partitional, then

- *K5 (Axiom of transparency) $K(E) \subseteq K(K(E))$.*
- *K6 (Axiom of wisdom) $\neg K(E) \subseteq K(\neg K(E))$, where $\neg K(E) = \Omega \backslash K(E)$.*

Note on Interpretation

1. *K1: Regardless of the actual state, the agent knows that he is in some state.*
2. *K2: If the agent knows E and knows F, then he knows $E \cap F$.*
3. *K3: If F occurs whenever E occurs, then knowing F means knowing E as well. This property can be derived directly from K2.*
4. *K4: If the agent knows E, then E must have occurred.*
5. *K5: If the agent knows E, then he knows that he knows E. Moreover, if $P(.)$ is partitional, then we can say $K(E) = E$ (and also $K(E) = K(K(E))$).*
6. *K6: If the agent doesn't know E, then he knows that he doesn't know E.*

Example 0.3 Puzzle of hats Each of three individuals is wearing a hat that is either black or white. Each can see others' hats, but not his own. The states of the world are

	a	b	c	d	e	f	g	h
1	B	B	B	B	W	W	W	W
2	B	B	W	W	B	B	W	W
3	B	W	B	W	B	W	B	W

.

And the information partitions are

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, h\}\}$$
$$P_2 = \{\{a, c\}, \{b, d\}, \{e, g\}, \{f, h\}\}.$$
$$P_3 = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}\}$$

Suppose that all the hats are black. If you ask any one whether he can identify the color of his own hat, the answer is always negative. Now, if you tell them that there is at least one black hat, the answer may change. The event that i knows the color of his own hat is $K_c^i := \{w : P_i(w) \in E_W^i \quad \text{or} \quad P_i(W) \in E_B^i\}$, and at the begining, $K_C^i = \emptyset$. Firstly, we can exclude h.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \not{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, d\}, \{e, g\}, \{f, \not{h}\}\}$$
$$P_3 = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, \not{h}\}\}$$

Then for player 1, we have $K_c^1 = \{d\}$ and 1 knows the answer if state is d, if 1 say no, it means the case cannot be d.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \not{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, \not{d}\}, \{e, g\}, \{f, \not{h}\}\}$$
$$P_3 = \{\{a, b\}, \{c, \not{d}\}, \{e, f\}, \{g, \not{h}\}\}$$

Then for player 2, 2 knows the answer if state is b or f, if player 2 say no, it cannot be b or f.

$$P_1 = \{\{a, e\}, \{b, f\}, \{c, g\}, \{d, \not{h}\}\}$$
$$P_2 = \{\{a, c\}, \{b, \not{d}\}, \{e, g\}, \{f, \not{h}\}\}$$
$$P_3 = \{\{a, \not{b}\}, \{c, \not{d}\}, \{e, \not{f}\}, \{g, \not{h}\}\}$$

1.2 Common Knowledge

Definition 1.6 (Common knowledge 1)

Let K_1 and K_2 be the knowledge functions of individuals 1 and 2 for the set Ω of states. An event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if ω is a member of every set in the infinite sequence $K_1(E), K_2(E), K_1(K_2(E))$, ...

Note on Interpretation *That is, at the state ω, 1 knows E, 2 knows E, 1 knows that 2 knows E, ...*

Definition 1.7 (Self-evident event)

Let P_1 and P_2 be the information functions of individuals 1 and 2 for the set Ω of states. An event $F \subseteq \Omega$ is self-evident between 1 and 2 if for all $\omega \in F$ we have $P_i(\omega) \subseteq F$ for $i = 1, 2$.

Note on $\neg F$ *If F is self-evident, then $\neg F$ is also self-evident.*

Note on Ω *The entire space is always self-evident and common knowledge.*

Definition 1.8 (Common knowledge 2)

An event $E \subseteq \Omega$ is common knowledge between 1 and 2 in the state $\omega \in \Omega$ if there is a self-evident event F for which $\omega \in F \subseteq E$.

Lemma 1.2 (Equivalence of Self-evident)

The following are equivalent:

1. $K_i(E) = E$ *for all* i,
2. E *is self-evident,*
3. *For all* i, E *is a union of members of the partition induced by* h_i.

Lemma 1.3

Definition 1.2 and Definition 1.2 are equivalent.

Note on Example *Let* $\Omega = \{a, b, c, d, e, f\}$ *and* $E = \{a, b, c, d\}$, *and*

$$\begin{aligned} P_1 &= \{\{a, b\}, \{c, d, e\}, \{f\}\} \\ P_2 &= \{\{a\}, \{b, c, d\}, \{e\}, \{f\}\} \end{aligned}.$$

Then we have

$$\begin{aligned} K_1(E) &= \{a, b\} \\ K_2(E) &= E \\ K_1(K_2(E)) &= \{a, b\} \\ K_2(K_1(E)) &= \{a\} \\ K_1(K_2(K_1(E))) &= \phi \end{aligned}$$

Thus E cannot be common knowledge between 1 and 2.

Definition 1.9 (Posterior belief)

Suppose that individuals 1 and 2 have the same prior belief $\rho(.)$ on Ω and partitional information functions $P_i(.)$. Then in some states $\omega^* \in \Omega$, individual i's posterior belief that some state in the event E has occurred is

$$\rho(E \mid P_i(\omega^*)) = \frac{\rho(E \cap P_i(\omega^*))}{\rho(P_i(\omega^*))}$$

Definition 1.10 (Knowledge function 2)

Individual i knows event E in state ω ($P_i(\omega) \subseteq E$) is equivalent to

$$\rho(E \mid P_i(\omega)) = 1$$

and then the knowledge function can be defined as

$$K_i(E) = \{\omega \in \Omega : \rho(E \mid P_i(\omega)) = 1\}.$$

Note on *More generally, we can define the event that individual i assigns probability q_i to event*

E as

$$\{\omega \in \Omega : \rho\,(E \mid P_i(\omega)) = q_i\}\,.$$

Note on Example *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\rho\,(\omega_1) = \rho\,(\omega_4) = 1/6$ and $\rho\,(\omega_2) = \rho\,(\omega_3) = 1/3$.*

$$P_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$$

$$P_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$$

Let $E = \{\omega_2, \omega_3\}$. It is easy to see that

$$\rho\,(E \mid P_i\,(\omega_s)) = \frac{\rho(w_2)}{\rho(w_1) + \rho(w_2)} = 2/3$$

$$\{\omega \in \Omega : \rho\,(E \mid P_i(\omega)) = 2/3\} = \Omega.$$

1.3 The Agreement Theorem

Theorem 1.1 (Agreeing to Disagree 1 (Aumann, 1976))

If two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors are equal.

Theorem 1.2 (Agreeing to Disagree 2 (Aumann, 1976))

Suppose two agents have the same prior belief over a finite set of states Ω. If each agent's information function is partitional and it is common knowledge in some state $\omega \in \Omega$ that agent 1 assigns probability η_1 to some event E and agent 2 assigns probability η_2 to E, then $\eta_1 = \eta_2$.

Note on Interpretation *Could two individuals who share the same prior over agree to disagree? That is, if i and j share a common prior over states, could a state arise at which it was commonly known that i assigned probability η_i to some event, j assigned probability η_j to that some event and $\eta_i \neq \eta_j$. This theorem concluded that this sort of disagreement is impossible.*

Note on Agreement Theorem to No-trade Theorem *Consider a trade where two agents bet on a coin, and this trade holds only when agent believe that $\Pr(\textit{ Heads }) > 1/2$ and agent believe that $\Pr(\textit{ Heads }) < 1/2$. However, Aumann's theorem says the bet cannot happen since these opposing beliefs would then be common knowledge!*

Proof Let p be a probability measure on Ω – interpreted as the agent's prior belief. For any state ω and event E, let $p\,(E \mid h_i(\omega))$ denote i's posterior belief, then the event that "i assigns probability η_i to E" is $\{\omega \in \Omega : p\,(E \mid h_i(\omega) = \eta_i\}$. By the assumption of common knowledge, there is some self-evident event F with $\omega \in F$ such that:

$$F \subset \{\omega' \in \Omega : p\,(E \mid h_1\,(\omega') = \eta_1\} \cap \{\omega' \in \Omega : p\,(E \mid h_2\,(\omega') = \eta_2\}$$

By Lemma 1.2, F is a union of members of i's information partition, i.e. $F = \cup_k A_k = \cup_k B_k$ (Ω is finite), where A and B are 1's and 2's information partition functions. Now, for any nonempty disjoint sets C, D with $p(E \mid C) = \eta_i$ and $p(E \mid D) = \eta_i$, since

$$p(E \mid C) = \frac{p(E \cap C)}{p(C)} = p(E \mid D) = \frac{p(E \cap D)}{p(D)} = \eta_i$$

$$p(E \mid C \cup D) = \frac{p(E \cap (C \cup D))}{p(C \cup D)} = \frac{p(E \cap C) + p(E \cap D)}{p(C) + p(D)} = \frac{\eta_i (p(C) + p(D))}{p(C) + p(D)},$$

we have $p(E \mid C \cup D) = \eta_i$. Then for each k, we have $p(E \mid F) = p(E \mid A_k) = \eta_1 = \eta_1$ and $p(E \mid F) = p(E \mid B_k) = \eta_2$. ■

Note on Example *Let $\Omega = \{a, b, c, d\}$ and each state occurs with prob 1/4, and*

$$P_1 = \{\{a, b\}, \{c, d\}\} \text{ and } P_2 = \{\{a, b, c\}, \{d\}\}.$$

Let $E = \{a, d\}$, since E is not common knowledge, at a we have

$$\begin{aligned} \eta_1(E) &= \rho(E \mid \{a, b\}) = 1/2 \\ \eta_2(E) &= \rho(E \mid \{a, b, c\}) = 1/3 \end{aligned}.$$

Player 1 knows $\eta_2(E)$, player 2 knows $\eta_1(E)$, but player 2 does not know what player 1 thinks of $\eta_2(E)$.

1.4 The No-Trade Theorem

Let Ω be a set of states and X a set of consequences (trading outcomes). A contingent contract is a function mapping Ω into X. Let A be the space of contracts. Each agent has a utility function $u_i : X \times \Omega \to \mathbb{R}$. Let $U_i(a) = u_i(a(\omega), \omega)$ denote i's utility from contract a – $U_i(a)$ is a random variable that depends on the realization of ω. Let $\mathbb{E}[U_i(a) \mid H_i]$ denote i's expectation of $U_i(a)$ conditional on his information H.

Theorem 1.3 (No-Trade Theorem 1 (Milgrom and Stokey (1982))

If a contingent contract b is ex ante efficient, then it cannot be common knowledge between the agents that every agent prefers contract a to contract b.

Theorem 1.4 (No-Trade Theorem 2 (Milgrom and Stokey (1982))

If ex ante allocation is Pareto optimal, then even after the players receive their private information, it cannot be common knowledge that they all expect to gain from trade.

1.5 E-mail Game (Rubinstein, 1989)

Even if each player is quite certain about the game being play, even small uncertainty about other's information can eliminate equilibria that exist when payoffs are common knowledge. Formally, the fact that small perturbations of the information structure can eliminate Nash equilibria occurs because the Nash equilibrium correspondence is not lower semi-continuous.

Consider the following Bayesian game, where $L > M > 1$. Player 1 is informed about the true game, 2 is not. The unique bayesian nash equilibrium is ((A,A),A).

If player 1 can communicate with player 2 in such a way that the true game becomes common

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

Figure 1: G_1 (1-p)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

Figure 2: G_2 (p<1/2)

knowledge, then there is a BNE in which both choose A in G_1, and both choose B in G_2. If the communication is imperfect, e.g. in G_2 player 1 sends a message to 2, and 2 sends back a confirmation message. With probability $\varepsilon > 0$, a message is not received, and each player sees the number of messages that he sent.

Formally, this bayesian game are

1. $\Omega = \{(k_1, k_2) : k_1 = k_2 \text{ or } k_1 = k_2 + 1\}$;
2. Signal function $\tau_i : \tau_i(k_1, k_2) = k_i$;
3. Common prior on Ω:

$$
\begin{aligned}
P_i(0,0) =& 1 - p \\
P_i(1,0) =& p\varepsilon \\
P_i(1,1) =& p\varepsilon(1-\varepsilon), \\
& \cdots \\
P_i(k+1,k) =& p\varepsilon(1-\varepsilon)^{2k} \\
P_i(k+1,k+1) =& p\varepsilon(1-\varepsilon)^{2k+1}
\end{aligned}
$$

Lemma 1.4

The e-mail game has a unique BNE, in which both players always choose A.

Proof

■

2 Solution Concept 4: Bayesian Nash Equilibrium

Definition 2.1 (Complete vs. Incomplete Information)

A complete information game is one where all players' payoff functions (and all other aspects of the game) are common knowledge.

Definition 2.2 (Common Belief)

All players share the same belief: $p_i = p(t) = p(t_1, t_2, \ldots, t_n)$ *for all* $i \in N$.

Definition 2.3 (Reduced form Bayesian Game (Li, 2022))

$$G = \{A_i; T_i; p_i; u_i\}_{i=1}^n$$

A reduced-form Bayesian game is a list as above, where

1. A_i *is the action space of* i,
2. T_i *is the type space of player* i, *and a strategy of a player* i *is any function* $s_i : T_i \to A_i$,
3. $p_i(t_{-i} \mid t_i)$ *is* i*'s belief about the other players,*
4. $u_i(a_1, \ldots, a_n; t_1, \ldots, t_n)$ *is* i*'s payoff function.*

Definition 2.4 (Reduced form Bayesian Nash equilibrium (Li, 2022))

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ *is a Bayesian Nash equilibrium (BNE) if for all* t_i, $s_i^*(t_i)$ *is a best response to* s_{-i}^*, *i.e.,* $s_i^*(t_i)$ *solves*

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) \cdot u_i\left(a_i, s_{-i}^*(t_{-i}); t\right).$$

Theorem 2.1 (Purification Theorem (Harsanyi 1973))

The mixed-strategy NE of a strategic game can be viewed as the limit of pure-strategy BNE in the slightly perturbed games.

Example 0.1 The stag hunt game has a mixed NE (p=q=1/2).

	Hare	Stag
Hare	$1, 1$	$2, 0$
Stag	$0, 2$	$3, 3$

Consider a perturbed game, where x and y are i.i.d. with uniform on $[-\varepsilon, \varepsilon]$, x and y are privately known by 1 and 2 respectively.

	Hare	Stag
Hare	$1 + x, 1 + y$	$2 + x, 0$
Stag	$0, 2 + y$	$3, 3$

Then there is a pure strategy BNE $(s_1(x), s_2(y))$ and beliefs:

$$s_1(x) = H \quad \text{iff } x > 0 \quad \text{and} \quad \Pr(s_2(y) = H|x) = \Pr(y > 0) = 1/2$$

$$s_2(y) = H \quad \text{iff } y > 0 \quad \text{and} \quad \Pr(s_1(x) = H|y) = \Pr(x > 0) = 1/2$$

And we can verify that the expected payoff of player 1 from choosing H is higher iff $x > 0$, so as player 2.

$$u_1(H|x) = \frac{1}{2}(1 + x) + \frac{1}{2}(2 + x) > u_1(S|x) = \frac{3}{2}$$

Definition 2.5 (Bayesian Game (Li, 2022))

A bayesian game consists of

1. *a finite set of players N, for each player $i \in N$,*
   - *a set of actions A_i,*
   - *a finite set of signals T_i (type space) and a signal function $\tau_i : \Sigma \to T_i$,*
   - *a probability distribution (prior belief) p_i on Σ, for which $p_i\left(\tau_i^{-1}(t_i)\right) > 0$ for all $t_i \in T_i$,*
   - *a vNM preference $\succeq_i$ over the set of lotteries over $A \times \Sigma$ (or equivalently, a Bernoulli payoff function $u_i(a; w)$);*
2. *a finite set of states Σ.*

Note on Strategy *In a Bayesian game, a strategy of a player i is any function $s_i : T_i \to A_i$.*

Definition 2.6 (Bayesian Nash equilibrium (Li, 2022))

A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Bayesian Nash equilibrium (BNE) if for all i and t_i, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{\omega \in \Omega} P_i(\omega \mid t_i) \cdot u_i\left(a_i, s_{-i}^*(\tau_{-i}(\omega)); \omega\right),$$

where the posterior belief is given by Bayes' law:

$$P_i(\omega \mid t_i) = \frac{p_i(\omega)}{p\left(\tau_i^{-1}(t_i)\right)} \text{ if } \omega \in \tau_i^{-1}(t_i)$$

$$P_i(\omega \mid t_i) = 0 \text{ if } \omega \notin \tau_i^{-1}(t_i)$$

Definition 2.7 (Separating, Pooling equilibrium)

3 Examples of finding BNE

3.1 Cournot Game with Incomplete information

3.2 Battle of the Sexes with Incomplete information

Suppose player 2 has two types and player 1 does not know player 2's type, but holds a belief about it. Then the expected payoffs are as follows, and the pure strategy NE is (F,(F,M)).

	F	M
F	$2, 1$	$0, 0$
M	$0, 0$	$1, 2$

Figure 3: Prob 1/2: 2 wishes to meet 1

	F	M
F	$2, 0$	$0, 2$
M	$0, 1$	$1, 0$

Figure 4: Prob 1/2: 2 wishes to avoid 1

	F, F	F, M	M, F	M, M
F	$\underline{2}, \frac{1}{2}$	$\underline{1}, \frac{3}{\underline{2}}$	$\underline{1}, 0$	$0, 1$
M	$0, \frac{1}{2}$	$1/2, 0$	$1/2, \frac{3}{\underline{2}}$	$\underline{1}, 1$

Note that in this example (reduced form), we do not elaborate the signal function, that is, we assume that player 2 also knows player 1 knows the state 1 with prob 1/2 and state 2 with prob 1/2. This assumption will be relaxed in the following sections.

3.3 Public good provision

Two players decide simultaneously whether or not to contribute to a public good. Each player i derives a commonly known value v if at least one of them contributes and 0 if none of them does. Player i's cost of contribution is c_i, only known to himself. It is common knowledge that c_1 and c_2 are $i.i.d.$ on $[c_L, c_H]$, and the continuous distribution function is $F(.)$. The reduced-form Bayesian game is:

1. Two players, $i = 1, 2$.
2. Player i's action space: $\{0, 1\}$, where 1 stands for "contribute", and 0 stands for "don't".
3. Player i's type space: $[c_L, c_H]$.
4. Player i's belief: $\Pr(c_j \leq c | c_i) = F(c)$.
5. Player i's payoff function:

$$u_i(a_1, a_2; c_1, c_2) = \begin{cases} 0 & \text{if } a_1 = a_2 = 0 \\ v - c_i & \text{if } a_i = 1 \\ v & \text{if } a_i = 0 \text{ and } a_j = 1 \end{cases}$$

Symmetric BNE: Assume that $v = 2$, c_1 and c_2 are uniformly distributed on $[1, 3]$. There is a symmetric BNE, in which $s_i^*(c_i) = 1$ iff $c_i \leq c^*$. To find c^*, note that BNE exhibits a monotonicity property and player i of type c^* must be indifferent between two actions. If $s_i(c_i) = 1$, his payoff is $2 - c^*$; if $s_i(c_i) = 0$, then his expected payoff is $\Pr(c_j \leq c^*) \cdot v = \frac{c^*-1}{3-1} \cdot 2 = c^* - 1$. Solve this equation, we obtain $c^* = 3/2$. The insight here is the game becomes inefficient if $c_i > 3/2$, even if both players find it profitable, there is no public good. Because both players have incentive to free ride.

$$2 - c^* = c^* - 1$$

Two Asymmetric BNE: in which one player never contributes and the other player contributes for all $c \leq v = 2$. The existence of such asymmetric equilibria depends on the common value v and the distribution of c_i. More specifically, asymmetric equilibria exist when

$$vF(v) \geq v - c_L$$

that is, it is optimal for a player with the lowest cost not to contribute if he believes that the other player contributes whenever that player's cost does not exceed v. For example, if $v = 1$ and $c_1, c_2 \sim U[0, 2]$, then there does not exist such asymmetric equilibria.

3.4 BoS with incomplete information and uncommon belief

	F	M
F	$2, 1$	$0, 0$
M	$0, 0$	$1, 2$

Figure 5: Prob 1/2: 2 wishes to meet 1

	F	M
F	$2, 0$	$0, 2$
M	$0, 1$	$1, 0$

Figure 6: Prob 1/2: 2 wishes to avoid 1

When player 2 wishes to meet player 1, she believes that with probability 2/3 player 1 knows it and with probability 1/3 he does not know; when player 2 wishes to avoid player 1, she believes that with probability 1/3 player 1 knows it and with probability 2/3 he does not know. Everything above is common knowledge between 1 and 2. The bayesian game thus can be formally formulated as below:

1. The set of players: $\{1, 2\}$;
2. The set of actions for each player i: $\{\mathrm{F}, \mathrm{M}\}$;
3. The set of states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where
   - in ω_1, player 2 wishes to meet player 1 and player 1 knows it,
   - in ω_2, player 2 wishes to meet player 1 and player 1 does not know it,
   - in ω_3, player 2 wishes to avoid player 1 and player 1 knows it,
   - in ω_4, player 2 wishes to avoid player 1 and player 1 does not know it.
4. For each player i, a set of signals T_i and a signal function $\tau_i : \Omega \to T_i$,
   - $T_1 = \{t_{11}, t_{12}, t_{13}\} : \tau_1(\omega_1) = t_{11}, \tau_1(\omega_3) = t_{12}$, and $\tau_1(\omega_2) = \tau_1(\omega_4) = t_{13}$.
   - $T_2 = \{t_{21}, t_{22}\} : \tau_2(\omega_1) = \tau_2(\omega_2) = t_{21}$, and $\tau_2(\omega_3) = \tau_2(\omega_4) = t_{22}$.
5. The common prior belief p on Ω: $p(\omega_1) = 1/3, p(\omega_2) = 1/6, p(\omega_3) = 1/6, p(\omega_4) = 1/3$.
6. Updated beliefs after receiving signals:
   - Player 1: since only when the state is ω_1, player 1 can receive signal t_{11}, player 1's updated belief after receiving t_{11} is:
   $$P_1(\omega_1 \mid t_{11}) = \frac{p(\omega_1)}{p\left(\tau_1^{-1}(t_{11})\right)} = \frac{p(\omega_1)}{p(\omega_1)} = 1 \quad \text{and} \quad P_1(\omega_s \mid t_{11}) = 0 \text{ for } \omega_s \neq \omega_1$$
   similarly, player 1's updated belief after receiving t_{12} is
   $$P_1(\omega_3 \mid t_{12}) = 1 \quad \text{and} \quad P_1(\omega_s \mid t_{12}) = 0 \text{ for } \omega_s \neq \omega_3$$
   and after receiving t_{13}, player 1 knows that the state is either ω_2 or ω_4, and using the Bayes rule, his updated belief is: $P_1(\omega_1 \mid t_{13}) = P_1(\omega_3 \mid t_{13}) = 0$, and
   $$P_1(\omega_2 \mid t_{13}) = \frac{p(\omega_2)}{p\left(\tau_1^{-1}(t_{13})\right)} = \frac{p(\omega_2)}{p(\omega_2) + p(\omega_4)} = \frac{1}{3}, P_1(\omega_4 \mid t_{13}) = \frac{2}{3}.$$
   - Player 2: after receiving t_{21}, player 2 knows that the state is either ω_1 or ω_2, and using the Bayes rule, her updated belief is: $P_2(\omega_3 \mid t_{21}) = P_2(\omega_4 \mid t_{21}) = 0$, and
   $$P_2(\omega_1 \mid t_{21}) = \frac{p(\omega_1)}{p\left(\tau_2^{-1}(t_{21})\right)} = \frac{p(\omega_1)}{p(\omega_1) + p(\omega_2)} = \frac{2}{3}, P_2(\omega_2 \mid t_{21}) = \frac{1}{3};$$

similarly, after receiving t_{22}, her updated belief is: $P_2(\omega_1 \mid t_{22}) = P_2(\omega_2 \mid t_{22}) = 0$,

$$P_2(\omega_3 \mid t_{22}) = \frac{p(\omega_3)}{p\left(\tau_2^{-1}(t_{22})\right)} = \frac{p(\omega_3)}{p(\omega_3) + p(\omega_4)} = \frac{1}{3}, P_2(\omega_4 \mid t_{22}) = \frac{2}{3}.$$

7. The vNM payoff functions:
   - Player 1: for all $\omega_s \in \Omega$,

     $$u_1((F,F);\omega_s) = 2, u_1((M,M);\omega_s) = 1, u_1((F,M);\omega_s) = u_1((M,F);\omega_s) = 0$$

   - Player 2:

     $$u_2((M,M);\omega_1) = u_2((M,M);\omega_2) = 2, u_2((F,F);\omega_1) = u_2((F,F);\omega_2) = 1$$

     $$u_2((F,M);\omega_1) = u_2((F,M);\omega_2) = u_2((M,F);\omega_1) = u_2((M,F);\omega_2) = 0$$

     and

     $$u_2((F,M);\omega_3) = u_2((F,M);\omega_4) = 2, u_2((M,F);\omega_3) = u_2((M,F);\omega_4) = 1$$

     $$u_2((F,F);\omega_3) = u_2((F,F);\omega_4) = u_2((M,M);\omega_3) = u_2((M,M);\omega_4) = 0$$

8. A strategy for player i is a function $s_i : T_i \to \{\text{F}, \text{M}\}$. That is, the strategy specifies the actions that player i chooses after receiving each possible signal.
9. Bayesian Nash equilibrium: a strategy profile s^* is a BNE if for each player i and each $t_i \in T_i$, $s_i^*(t_i)$ solves

   $$\max_{a_i \in \{F,M\}} \sum_{\omega_s \in \Omega} P_i(\omega_s \mid t_i) \cdot u_i\left(a_i, s_j^*(\tau_2(\omega_s)); \omega_s\right)$$

   That is, after receiving a signal $t_i \in T_i$, the action $s_i^*(t_i) \in \{\text{F}, \text{M}\}$ gives player i the highest expected payoff calculated using his updated beliefs.

Note that player 1's strategy consists of three actions and player 2's strategy consists of two actions, one action for each signal that the player may receive. Thus, player 1 has eight pure strategies and player 2 has four pure strategies. In our analysis below, we consider the possibility that each of player 2's pure strategies is used in a BNE.

1. FF: If player 2 always chooses F in a BNE, then player 1's strategy in such a BNE must be choosing F regardless of the signal he receives. However, if player 1 always chooses F, it is not optimal for player 2 to choose F after receiving signal t_{22}, i.e. player 2 knows that the state is either ω_3 or ω_4, in which she receives a higher payoff from action profile (F,M) than from (F,F). Thus FF can not be player 2's strategy in a BNE.
2. MM: Similarly, MM cannot be player 2's strategy in a BNE.
3. MF, i.e. choosing M (F resp.) after receiving t_{21} (t_{22} resp.). It is easy to see that player 1's best response to this strategy is to choose M after receiving t_{11}, choose F after receiving t_{12}, and choose F after receiving t_{13}. Then we need to check whether MF is player 2's best response to player 1's strategy MFF. When player 2 receives t_{22}, her updated belief is that with probability 1/3, the state is ω_3, in which player 1 would receive signal t_{12} and choose F; with probability 2/3, the state is ω_4, in which player 1 would receive signal t_{13} and also choose F. Clearly, it cannot be optimal for player to choose F. Thus, MF cannot be player 2's strategy in a BNE.

- To identify player 1's optimal action after receiving t_{13}, we need to calculate player 1's expected payoff from using either action. Note that after receiving t_{13}, player 1's update belief is that with probability 1/3, the state is ω_2, in which player 2 would receive signal t_{21} and choose M; with probability 2/3, the state is ω_4, in which player 2 would receive signal t_{22} and choose F. Thus, player 1's expected payoff from choosing F is $2 \times 2/3 = 4/3$, his expected payoff from choosing M is $1 \times 1/3 = 1/3$, and it is optimal to choose F.

4. FM, i.e., choosing F (M resp.) after receiving t_{21} (t_{22} resp.) : It is easy to see that player 1's best response to this strategy is to choose F after receiving t_{11}, choose M after receiving t_{12}, and player 1 is indifferent between F and M after receiving t_{13}, both FMM and FMF are player 1's best responses to player 2's strategy FM. Next we want to check whether MF is player 2's best response to FMM or FMF, and only (FMF, FM) is a BNE.
   - Note that after receiving t_{13}, player 1's update belief is that with probability 1/3, the state is ω_2, in which player 2 would receive signal t_{21} and choose F; with probability 2/3, the state is ω_4, in which player 2 would receive signal t_{22} and choose M. Thus, player 1's expected payoff from choosing F is $2 \times 1/3 = 2/3$, his expected payoff from choosing M is $1 \times 2/3 = 2/3$, and it is optimal to choose F.
   - FMM: After receiving t_{22}, player 2's updated belief is that the state is either $\omega(1/3)$ or $\omega_4(2/3)$, in which player 1 receives either t_{12} or t_{13}. Since player 1 will choose M after receiving t_{12} or t_{13}, it is not optimal for player 2 to choose M after receiving t_{22}. Thus, (FMM, FM) is not a BNE.
5. FMF: After receiving t_{21}, player 2's updated belief is that the state is either $\omega_1(2/3)$ or $\omega_2(1/3)$, player receives either t_{11} or t_{13}. Since player 1 would choose F after receiving t_{11} or t_{13}, it is optimal for player 2 to choose F after receiving t_{21}. After receiving t_{22}, player 2's updated belief is that the state is either $\omega_3(1/3)$ or $\omega_4(2/3)$, player 1 receives either t_{12} or t_{13}, and choose F (M resp.) with probability 2/3 (1/3 resp.), it can be verified that M is indeed the optimal action for player 2 after receiving t_{22}. Thus, (FMF, FM) is a BNE.

To sum up, there is a unique pure-strategy BNE:

$$s_1^*(t_{11}) = F, s_1^*(t_{12}) = M, s_1^*(t_{13}) = F; s_2^*(t_{21}) = F, s_2^*(t_{22}) = M.$$

Note on Existence *In general, a pure-strategy BNE may not exist, but Nash existence theorem still applies, that is, a finite Bayesian game always has a BNE in pure or mixed strategies.*

Note on Reduced form *This game can also be analyzed in its reduced form the same as we do under the state-space formulation. While the later gives us a better understanding of the concept of "type".*

1. *The set of players:* $\{1, 2\}$*;*
2. *The set of actions for each player* i*:* $\{\mathrm{F}, \mathrm{M}\}$*;*
3. *For each player* i*, a set of signals* T_i *and a signal function* $\tau_i : \Omega \to T_i$*,*

- $T_1 = \{t_{11}, t_{12}, t_{13}\} : \tau_1(\omega_1) = t_{11}, \tau_1(\omega_3) = t_{12}$, *and* $\tau_1(\omega_2) = \tau_1(\omega_4) = t_{13}$.
- $T_2 = \{t_{21}, t_{22}\} : \tau_2(\omega_1) = \tau_2(\omega_2) = t_{21}$, *and* $\tau_2(\omega_3) = \tau_2(\omega_4) = t_{22}$.

4. *Beliefs*
   - *Player 1*

$$P_1(t_{21} \mid t_{11}) = 1, P_1(t_{22} \mid t_{11}) = 0$$
$$P_1(t_{21} \mid t_{12}) = 0, P_1(t_{22} \mid t_{12}) = 1$$
$$P_1(t_{21} \mid t_{13}) = \frac{1}{3}, P_1(t_{22} \mid t_{13}) = \frac{2}{3}$$

   - *Player 2*

$$P_2(t_{11} \mid t_{21}) = \frac{2}{3}, P_2(t_{12} \mid t_{21}) = 0, P_2(t_{13} \mid t_{21}) = \frac{1}{3}$$
$$P_2(t_{11} \mid t_{21}) = 0, P_2(t_{12} \mid t_{21}) = \frac{1}{3}, P_2(t_{13} \mid t_{21}) = \frac{2}{3}$$

5. *The vNM payoff functions:*
   - *Player 1: for all type profile* $\mathbf{t} \in T_1 \times T_2$*, i.e., player 1's payoff depends only on the action profile, not on the types.*

$$u_1((F,F);\mathbf{t}) = 2, u_1((M,M);\mathbf{t}) = 1, u_1((F,M);\mathbf{t}) = u_1((M,F);\mathbf{t}) = 0$$

   - *Player 2: for all type profile* $\mathbf{t}_{21} \in T_1 \times \{t_{21}\}$

$$u_2((M,M);\mathbf{t}_{21}) = 2, u_2((F,F);\mathbf{t}_{21}) = 1, u_2((F,M);\mathbf{t}_{21}) = u_2((M,F);\mathbf{t}_{21}) = 0$$

   and for all type profile $\mathbf{t}_{22} = T_1 \times \{t_{22}\}$

$$u_2((F,M);\mathbf{t}_{22}) = 2, u_2((M,F);\mathbf{t}_{22}) = 1, u_2((F,F);\mathbf{t}_{22}) = u_2((M,M);\mathbf{t}_{22}) = 0$$

4 Auction Theory

4.1 Auction Mechanism

Definition 4.1 (Open ascending auction (English auction))

Definition 4.2 (Open descending auction (Dutch auction))

Definition 4.3 (First-Price (Sealed Bid) Auction)

Definition 4.4 (Second-Price (Sealed Bid) Auction)

Lemma 4.1 (Strategic equivalence of different auctions)

Definition 4.5 (Auction Mechanism)

An auction is a mechanism with well-defined allocation rule and payment rule.

1. *Allocation rule:*
2. *Payment rule:*

Definition 4.6 (Assumptions for Auction)

1. *Independent Private Value: v_i are independent and private.*
2. *Symmetry: same distribution F for v_i*
3. *Zero-one support: F has a support of $[0, 1]$*
4. *Linear payoff: u(win)=$v_i - p_i$*
5. *Risk Neutrality:*

Definition 4.7 (Bidding function)

Bidder i's bid is a function from $[0, 1]$ to non-negative number: $\beta_i : [0, 1] \to \mathbb{R}_+$.

Definition 4.8 (Bidding Strategy)

Bidding strategy β_i can be represented by a graph.

1. *Truthful bidding:*
2. *Overbidding:*
3. *Underbidding (Bid-shading):*

Figure

4.2 Second-price auction (Kartik, 2009)

There are I bidders, with value $0 \leq v_1 \leq ... \leq v_I$, and their values are common knowledge. All bidders simultaneously bid $s_i \in [0, \infty]$, the highest one wins the auction and pays the second-high price. Define $W(s) = k : \forall j, s_k \geq s_j$ as the set of highest bidders, then bidder i's utility is

$$u_i(s_i, s_{-i}) = \begin{cases} v_i - max_{j \neq i} s_j & \text{if } s_i > max_{j \neq i} s_j \\ \frac{1}{|W(s)|}(v_i - s_i) & \text{if } s_i = max_{j \neq i} s_j \\ 0 & \text{if } s_i < max_{j \neq i} s_j \end{cases}$$

Note on Another mechanism under equality *When more than one bidder submits the highest bid, each gets the object with equal probability by a lottery, and the payment is equal to the highest bid in this case. Note that this does not change our results.*

Lemma 4.2 (BNE for Second-price Auction)

Everyone chooses to bid their real valuation $s_i = v_i$, and this strategy is weakly dominant.

Proof Let $m(s_{-i}) = max_{j \neq i} s_j$. Supposes $s_i > v_i$, then for any strategy profile s_{-i}, if $m(s_{-i}) > s_i$, then $u_i(s_i, s_{-i}) = 0$; if $m(s_{-i}) \leq v_i$, then $u_i(s_i, s_{-i}) = u_i(v_i, s_{-i}) \geq 0$;

otherwise if $m(s_{-i}) \in (v_i, s_i]$, then $u_i(s_i, s_{-i}) < 0$. Thus when $s_i > v_i$, $s_i = v_i$ is weakly dominant. The case of $s_i < v_i$ can be proved similarly. ■

4.3 First-price Auction

Lemma 4.3 (BNE for Uniform First-price Auction)

In a first-price auction, assuming values are i.i.d. uniformly distributed on $[0, \bar{v}]$, the bidding strategy $b_i = \left(\frac{n-1}{n}\right) v_i$ comprises a symmetric Bayesian Nash equilibrium.

Proof Fix a bidder i. We assume that all bidders chooses b_i according to linear strategy $s_i = \alpha v_i + \beta$, and argue that bidder i should do the same to find the value of α and β. Note that the probability of bidder i winning the auction is $Pr(s_i > s_j, j \neq i)$, and the expected utility is

$$E[s_i] = (\frac{s_i - \beta}{\alpha \bar{v}})^{N-1}(v_i - s_i)$$

By the FOC, we have $s_i = \frac{(N-1)v_i+\beta}{N}$. Combine the condition $s_i = \alpha v_i + \beta$, we have $\alpha = \frac{N-1}{N}$ and $\beta = 0$, and the bidding strategy $s_i^* = \frac{N-1}{N} v_i$ comprises a symmetric BNE. ■

4.4 Revenue Equivalence

Definition 4.9 (kth-order statistic)

The kth-order statistic, denoted $X_{(k)}$, is the kth-largest value among n draws of a random variable X.

Lemma 4.4 ($E[X_{(1)}]$ for $X \sim U[0,1]$)

$$E[X_{(1)}] = \int_0^1 x f_{X_{(1)}}(x) dx = \frac{n}{n+1}$$

Proof

$$\begin{aligned} F_{X_{(1)}}(x) &= \Pr\left(X_{(1)} \leq x\right) \\ &= \prod_n U(x) \\ &= x^n. \end{aligned}$$

■

Lemma 4.5 ($E[X_{(2)}]$ for $X \sim U[0,1]$)

$$E[X_{(2)}] = \int_0^1 x f_{X_{(2)}}(x) dx = \frac{n-1}{n+1}$$

Proof

$$F_{X_{(2)}}(x) = \Pr\left(X_{(2)} \leq x\right) = x^n + n x^{n-1}(1-x)$$

■

Theorem 4.1 (Revenue Equivalence)

If bidder's values are uniform $i.i.d.$, then the expected revenue of the first-price auction is equal to that of the second-price auction, assuming bidders behave according to their respective equilibrium strategies.

Proof The support of the uniform distribution does not matter; we choose $[0, 1]$ for convenience. Let R_1 and R_2 denote the expected revenue of the first- and second-price auctions, respectively.

$$R_2 = \frac{n-1}{n+1}$$

$$R_1 = \mathbb{E}\left[\left(\frac{n-1}{n}\right) v_{\max}\right] = \left(\frac{n-1}{n}\right) \mathbb{E}\left[v_{\max}\right] = \frac{n-1}{n+1}$$

■

4.5 Double Auction

Definition 4.10 (Double Auction)

Definition 4.11 (BNE for Double Auction)

Lemma 4.6 (Single Price BNE)

Lemma 4.7 (Linear BNE)

5 From Bayesian Game to Mechanism Design: The Revelation Principle

Theorem 5.1 (The Revelation Principle)

Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism.

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